Exploiting the Circulant Structure of Tracking-by-detection with Kernels

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Appendix C: Additional circulant matrix properties

This appendix will give more complete derivations of properties that, while easy enough to verify through direct computation, merit a more detailed analysis. These proofs were not included in the main paper to meet length requirements.

It also represents an attempt to better formalize the tools that were used in the main paper, and may be useful in other domains.

First we will formalize a definition given in Section 2.2. The fact that each row i of $C(\mathbf{u})$ is given by $P^i\mathbf{u}$ can be expressed in the following way.

Definition 1. *The rows of an $n \times n$ circulant matrix $C(\mathbf{u})$ are given by:*

$$C(\mathbf{u}) = \begin{bmatrix} \left(P^0\mathbf{u}\right)^T \\ \vdots \\ \left(P^{n-1}\mathbf{u}\right)^T \end{bmatrix}, \tag{25}$$

where P is the $n \times n$ permutation matrix that produces cyclic shifts of a vector,

$$P = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{26}$$

Property 1. **Transposed circulant matrices and correlation**

Transposing a circulant matrix induces complex-conjugation in the Fourier domain; ie.,

$$C^T(\mathbf{u}) = C(\mathcal{F}^{-1}(\mathcal{F}^*(\mathbf{u}))), \tag{27}$$

with * denoting the complex-conjugate. This suggests an alternative to Eq. 4, only with a transposed circulant matrix:

$$C^T(\mathbf{u})\mathbf{v} = \mathcal{F}^{-1}\left(\mathcal{F}(\mathbf{u}) \odot \mathcal{F}(\mathbf{v})\right). \tag{28}$$

Notice that the complex-conjugate was cancelled out. Eq. 28 can be understood as encoding correlation instead of convolution, which differ only by flipping the order of the elements of one of the input vectors.

Eq. 27 was used in Eq. 24 (Appendix A.2) to cancel out the complex-conjugation from the $\mathcal{F}(\bar{\mathbf{k}})$ term.

Property 2. **Symmetricity of the reduced kernel matrix representation**

This property establishes that the vector $\mathbf{k}$, which compactly represents the kernel matrix through $K = C(\mathbf{k})$ (Eq. 6), is symmetric and thus has real Fourier transform:

$$\mathcal{F}\left(\mathbf{k}\right) = \mathcal{F}^*\left(\mathbf{k}\right). \tag{29}$$

This property was used in Eq. 22 to remove the complex-conjugation from the $1/\left(\mathcal{F}\left(\mathbf{k}\right) + \lambda\right)$ term, which would arise from Eq. 4.

Proof. The kernel matrix K is symmetric ($K = K^T$), due to positive definiteness of the kernel κ. From the compact representation of the kernel matrix $K = C(\mathbf{k})$ (Eq. 6), and Eq. 27 in Property 1, we get

$$K^T = C^T(\mathbf{k}) = C(\mathcal{F}^{-1}(\mathcal{F}^*(\mathbf{k}))). \tag{30}$$

Comparing it to $K = C(\mathbf{k}) = C(\mathcal{F}^{-1}(\mathcal{F}(\mathbf{k})))$, we find that

$$\mathcal{F}\left(\mathbf{k}\right) = \mathcal{F}^*\left(\mathbf{k}\right). \tag{31}$$

As an aside, since $K = K^T$ we find that the vector $\mathbf{k}$ is also symmetric (its elements satisfy $k_i = k_{n-i}$). The fact that a symmetric signal has real Fourier transform is a well-known result from signal processing theory [21].

Property 3. **Matrix form of dot-product kernels**

This property was used in Section 3.1, and establishes the equivalence between the definition of the elements of vector $\mathbf{k}^{\mathrm{dp}}$

$$k_i^{\mathrm{dp}} = \kappa(\mathbf{x},\, P^i\mathbf{x}') = g\big(\mathbf{x}^T P^i \mathbf{x}'\big)\,,\ \forall i = 1, \ldots, n \tag{32}$$

and the same vector in matrix notation,

$$\mathbf{k}^{\mathrm{dp}} = g(C(\mathbf{x}')\,\mathbf{x})\,. \tag{33}$$

Proof. One way to prove this is to make some part of k_i^{dp} conform to Definition 1, thus constructing a circulant matrix. Because k_i^{dp} is a scalar,

$$k_i^{\mathrm{dp}} = g\Big(\big(P^i\mathbf{x}'\big)^T\,\mathbf{x}\Big)\,. \tag{34}$$

Concatenating the elements into a vector, and since $g(\cdot)$ is an element-wise function,

$$\mathbf{k}^{\mathrm{dp}} = \begin{bmatrix} g\Big(\big(P^0\mathbf{x}'\big)^T\mathbf{x}\Big) \\ \vdots \\ g\Big(\big(P^{n-1}\mathbf{x}'\big)^T\mathbf{x}\Big) \end{bmatrix} = g\left(\begin{bmatrix} \big(P^0\mathbf{x}'\big)^T\mathbf{x} \\ \vdots \\ \big(P^{n-1}\mathbf{x}'\big)^T\mathbf{x} \end{bmatrix}\right). \tag{35}$$

Taking the $\mathbf{x}$ out of the multiplication, and using Definition 1, we get

$$\mathbf{k}^{\mathrm{dp}} = g\left(\begin{bmatrix} \big(P^0\mathbf{x}'\big)^T \\ \vdots \\ \big(P^{n-1}\mathbf{x}'\big)^T \end{bmatrix}\mathbf{x}\right) = g(C\,(\mathbf{x}')\,\mathbf{x})\,. \tag{36}$$

Property 4. **Matrix form of kernel coefficients convolution**

This property was used in Section A.2 to prove Eq. 9, the fast detection formula. It consists of expressing Eq. 23 (reproduced here),

$$\hat{y}_i = \sum_j \alpha_j \kappa(P^i\mathbf{z},\, P^j\mathbf{x}),\ \forall i = 1,\ldots,n \tag{37}$$

in matrix form,

$$\hat{\mathbf{y}} = C^T(\bar{\mathbf{k}})\boldsymbol{\alpha}, \tag{38}$$

where $\bar{\mathbf{k}}$ is the vector with elements $\bar{k}_i = \kappa(\mathbf{z},\, P^i\mathbf{x})$, and $\boldsymbol{\alpha}$ is the vector with elements α_i.

Proof. We can build a kernel matrix $\bar{K}$ between shifted versions of $\mathbf{z}$ and shifted versions of $\mathbf{x}$, with elements

$$\bar{k}_{ij} = \kappa(P^i\mathbf{z},\, P^j\mathbf{x}). \tag{39}$$

Note that, unlike the kernel matrix K used for training, $\bar{K}$ is not necessarily symmetric, because $\mathbf{x}$ and $\mathbf{z}$ are different.

Then, the elements in Eq. 37 can be concatenated into a vector $\hat{\mathbf{y}}$, and expressed as the multiplication

$$\hat{\mathbf{y}} = \bar{K}^T\boldsymbol{\alpha}, \tag{40}$$

which can be verified to be equivalent to Eq. 37 by direct computation.

With the same argument as Theorem 1, we can see that $\bar{K}$ is circulant. As such, we can express it in reduced form as $\bar{K} = C(\bar{\mathbf{k}})$, with the vector $\bar{\mathbf{k}}$ as defined earlier. We then obtain

$$\hat{\mathbf{y}} = C^T(\bar{\mathbf{k}})\boldsymbol{\alpha}. \tag{41}$$

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